

# EXPLICIT INVERSIONS OF CERTAIN MATRICES II

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**ABSTRACT.** In this note, we apply kernel polynomials to find the explicit inverses for some some Hankel matrices associated with  $q$ -orthogonal polynomials.

## 1. INTRODUCTION

In the theory of orthogonal polynomials, We could calculate the determinants of some Hankel matrices once we know the three term recurrence relation for the associated orthogonal polynomials and vice versa. It is well-known that the kernel polynomials of the orthogonal polynomials encode important information about the associated Hankel matrices. These matrices are generalizations of the Hilbert matrices. In this note we invert some Hankel matrices associated with some  $q$ -polynomials by using the kernel polynomials.

The rest of the introduction could be found in [4], we copy it here for the convenience of the reader.

**Theorem 1.1.** *Given a probability measure  $\mu$  on  $\mathbb{R}$  with a support of infinite many points. Let us consider the Hilbert space of  $\mu$ -measurable functions*

$$(1.1) \quad \mathcal{X} := \left\{ f(x) \mid \int |f(x)|^2 d\mu(x) < \infty \right\}$$

*with the inner product defined as*

$$(1.2) \quad (f, g) := \int f(x) \overline{g(x)} d\mu(x), \quad f, g \in \mathcal{X}.$$

*Assume that  $\{w_n(x)\}_{n=0}^\infty$  is a sequence of linearly independent functions in  $\mathcal{X}$  with  $w_0(x) = 1$ . Let*

$$(1.3) \quad \alpha_{jk} := \int w_j(x) \overline{w_k(x)} d\mu(x), \quad j, k = 0, 1, \dots,$$

$$(1.4) \quad \Pi_n := \begin{pmatrix} \alpha_{00} & \alpha_{01} & \dots & \alpha_{0n} \\ \alpha_{10} & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n0} & \alpha_{n1} & \dots & \alpha_{nn} \end{pmatrix}, \quad n \in \mathbb{N} \cup \{0\},$$

*and*

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$$(1.5) \quad \Delta_n := \det \Pi_n, \quad n \in \mathbb{N} \cup \{0\}.$$

Then the  $n$ -th orthonormal function with positive coefficient in  $w_n(x)$  is given by the formula

$$(1.6) \quad p_n(x) = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \det \begin{pmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \dots & \alpha_{0n} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n0} & \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \\ w_0(x) & w_1(x) & w_2(x) & \dots & w_n(x) \end{pmatrix}$$

for  $n \in \mathbb{N}$ , with

$$(1.7) \quad p_0(x) = w_0(x) = 1.$$

Furthermore, the coefficient of  $p_n(x)$  in  $w_n(x)$  is

$$(1.8) \quad \gamma_n := \sqrt{\frac{\Delta_{n-1}}{\Delta_n}}.$$

*Proof.* The proof is the same as for the case  $w_n(x) = x^n$ , which could be found in any orthogonal polynomials textbooks such as [3].  $\square$

**Corollary 1.2.** For  $n \in \mathbb{N}$ , we have

$$(1.9) \quad \Delta_n = \prod_{k=1}^n \frac{1}{\gamma_k^2}.$$

*Proof.* This is a trivial consequence of (1.7) and (1.8).  $\square$

**Lemma 1.3.** Let

$$(1.10) \quad k_n(x, y) := \sum_{k=0}^n p_k(x) \overline{p_k(y)}, \quad n \in \mathbb{N} \cup \{0\}.$$

Then, for any  $\pi(x)$  in the linear span of  $\{w_k(x)\}_0^n$ , we have

$$(1.11) \quad \int \pi(x) \overline{k_n(x, y)} d\mu(x) = \pi(y).$$

*Proof.* To see (1.11), just expand  $\pi(x)$  in  $p_k(x)$ ,  $k = 0, 1, \dots, n$ .  $\square$

**Lemma 1.4.** For each  $n \in \mathbb{N} \cup \{0\}$ , the function  $k_n(x, y)$  satisfying (1.11) is unique.

*Proof.* Suppose there are two such functions  $h_n(x, y)$  and  $k_n(x, y)$ , then,

$$(1.12) \quad \begin{aligned} 0 &< \|h_n(\cdot, y) - k_n(\cdot, y)\|^2 \\ &= (h_n(\cdot, y) - k_n(\cdot, y), h_n(\cdot, y) - k_n(\cdot, y)) \\ &= (h_n(\cdot, y) - k_n(\cdot, y), h_n(\cdot, y)) - (h_n(\cdot, y) - k_n(\cdot, y), k_n(\cdot, y)) \\ &= 0, \end{aligned}$$

which is a contradiction.  $\square$

**Lemma 1.5.** *Let*

$$(1.13) \quad (\beta_{jk})_{0 \leq j, k \leq n} := \Pi_n^{-1}, \quad n \in \mathbb{N} \cup \{0\}.$$

*Then,*

$$(1.14) \quad k_n(x, y) = \sum_{j, k=0}^n \beta_{jk} \overline{w_j(y)} w_k(x).$$

*Proof.* Let

$$(1.15) \quad f(x) = \sum_{k=0}^n u_k w_k(x),$$

then,

$$(1.16) \quad \begin{aligned} & (f(\cdot), \sum_{j, k=0}^n \beta_{jk} \overline{w_j(y)} w_k(\cdot)) \\ &= \sum_{m=0}^n u_m (w_m(\cdot), k(\cdot, y)) \\ &= \sum_{m=0}^n u_m \sum_{j, k=0}^n \overline{\beta_{jk} w_j(y)} (w_m, w_k) \\ &= \sum_{m=0}^n u_m \sum_{j=0}^n w_j(y) \sum_{k=0}^n \overline{\beta_{jk} \alpha_{km}} \\ &= f(y). \end{aligned}$$

By Lemma 1.4, we have

$$(1.17) \quad k_n(x, y) = \sum_{j, k=0}^n \beta_{jk} \overline{w_j(y)} w_k(x).$$

□

**Corollary 1.6.** *The kernel in Lemma 1.3 is also given by*

$$(1.18) \quad k_n(x, y) = -\frac{1}{\Delta_n} \det \begin{pmatrix} 0 & 1 & \overline{w_1(y)} & \cdots & \overline{w_n(y)} \\ 1 & \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0n} \\ w_1(x) & \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_n(x) & \alpha_{n0} & \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}$$

for  $n \in \mathbb{N} \cup \{0\}$ .

*Proof.* Since

$$(1.19) \quad k_n(x, y) = \sum_{j, k=0}^n \beta_{jk} \overline{w_j(y)} w_k(x),$$

with

$$(1.20) \quad (\beta_{jk})_{0 \leq i, j \leq n} = \Pi_n^{-1}.$$

Then,

$$(1.21) \quad \beta_{jk} = \frac{\Pi_n(k, j)}{\det \Pi_n} = \frac{\Pi_n(k, j)}{\Delta_n},$$

where  $\Pi_n(k, j)$  is the  $(k, j)$ -th co-factor. Therefore,

$$(1.22) \quad k_n(x, y) = \frac{1}{\Delta_n} \sum_{j,k=0}^n \Pi_n(k, j) \overline{w_j(y)} w_k(x).$$

It is clear that

$$(1.23) \quad \sum_{j,k=0}^n \Pi_n(k, j) \overline{w_j(y)} w_k(x) = - \begin{vmatrix} 0 & (\overline{\mathbf{W}(\mathbf{y})})^T \\ \mathbf{W}(\mathbf{x}) & \Pi_n \end{vmatrix},$$

by direct determination expansion, which is

$$(1.24) \quad k_n(x, y) = -\frac{1}{\Delta_n} \begin{vmatrix} 0 & (\overline{\mathbf{W}(\mathbf{y})})^T \\ \mathbf{W}(\mathbf{x}) & \Pi_n \end{vmatrix},$$

where

$$(1.25) \quad \mathbf{W}(\mathbf{x}) = \begin{pmatrix} 1 \\ w_1(x) \\ \vdots \\ w_n(x) \end{pmatrix},$$

and

$$(1.26) \quad (\overline{\mathbf{W}(\mathbf{y})})^T = (1, \overline{w_1(y)}, \dots, \overline{w_n(y)}).$$

□

Lemma 1.5 enables us to compute the inverse the Gram matrix in terms of the orthonormal functions  $\{p_n(x)\}_{n=0}^\infty$ .

**Corollary 1.7.** *Assume that  $\{w_n(x)\}_{n=0}^\infty$ ,  $\{p_n(x)\}_{n=0}^\infty$  and  $\Pi_n = (\alpha_{jk})_{0 \leq j, k \leq n}$  as in Theorem 1.1. Suppose we have two families of linear functionals  $\{u_k\}_{k=0}^\infty$  and  $\{v_k\}_{k=0}^\infty$  over the linear space generated by  $\{w_n(x)\}_{n=0}^\infty$  with*

$$(1.27) \quad u_j(w_k) = \delta_{jk},$$

and

$$(1.28) \quad v_j(\overline{w_k}) = \delta_{jk}$$

for  $j, k = 0, 1, \dots$ . Then,

$$(1.29) \quad \beta_{jk} = \sum_{m=0}^n u_k(p_m(x)) v_j(\overline{p_m(y)}),$$

where

$$(1.30) \quad (\beta_{jk})_{0 \leq j, k \leq n} = \Pi_n^{-1}.$$

*Proof.* From Lemma 1.5, we have

$$(1.31) \quad \sum_{j,k=0}^n \beta_{jk} \overline{w_j(y)} w_k(x) = \sum_{m=0}^n \overline{p_m(y)} p_m(x).$$

Then we apply the functional  $u_j$  and  $v_k$  both sides of the above equation, the claim of the corollary follows.  $\square$

## 2. MAIN RESULTS

As usual,  $[1, 2]$ ,

$$(2.1) \quad (a; q)_\infty := \prod_{m=0}^{\infty} (1 - aq^m), \quad a \in \mathbb{C}, \quad q \in (0, 1),$$

and

$$(2.2) \quad (a; q)_m := \frac{(a; q)_\infty}{(aq^m; q)_\infty}, \quad m \in \mathbb{Z}.$$

We will use the following short hand notations

$$(2.3) \quad \left[ \begin{matrix} m \\ j \end{matrix} \right]_q := \frac{(q; q)_m}{(q; q)_j (q; q)_{m-j}}, \quad j \leq m, \quad j, m \in \mathbb{N} \cup \{0\},$$

and

$$(2.4) \quad (a_1, a_2, \dots, a_n; q)_m := \prod_{k=1}^n (a_k; q)_m, \quad n \in \mathbb{N}, \quad m \in \mathbb{Z}, \quad a_1, \dots, a_n \in \mathbb{C}.$$

The basic hypergeometric function  ${}_r\phi_s$  with complex parameters  $a_1, \dots, a_r; b_1, \dots, b_s$  is formally defined as

$$(2.5) \quad {}_r\phi_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q; z \right) := \sum_{m=0}^{\infty} \frac{(a_1, \dots, a_r; q)_m}{(q, b_1, \dots, b_s; q)_m} \left( (-1)^n q^{(n-1)n/2} \right)^{s+1-r} z^m.$$

The  $q$ -binomial theorem is the following summation formula

$$(2.6) \quad \frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n \quad |z| < 1,$$

and the Ramanujan  ${}_1\psi_1$  summation formula is

$$(2.7) \quad \sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(b/a, q, q/az, az; q)_\infty}{(b, b/az, q/a, z; q)_\infty}, \quad \left| \frac{b}{a} \right| < |z| < 1.$$

**Theorem 2.1.** For  $n \in \mathbb{N} \cup \{0\}$ , the matrix

$$(2.8) \quad \left( \frac{(aq; q)_{j+k}}{(abq^2; q)_{j+k}} \right)_{j,k=0}^n$$

has determinant

$$\begin{aligned}
(2.9) \quad & \det \left( \frac{(aq; q)_{j+k}}{(abq^2; q)_{j+k}} \right)_{j,k=0}^n \\
&= a^{n(n+1)/2} q^{n(n+1)(2n+1)/6} \prod_{k=1}^n \frac{(q, aq, bq, abq; q)_k}{(abq, abq^2, abq^2, abq^3; q^2)_k}.
\end{aligned}$$

The matrix (2.8) is invertible for

$$(2.10) \quad q^k, aq^k, bq^k, abq^k \neq 1, \quad k \in \mathbb{N}, \quad a, q \neq 0,$$

its inverse matrix  $(\gamma_{jk})_{j,k=0}^n$  has elements

$$\begin{aligned}
(2.11) \quad & \gamma_{jk} = \frac{(-1)^{j+k} q^{[j(j+1)+k(k+1)]/2}}{(aq; q)_j (aq; q)_k} \\
& \times \sum_{m=\max(j,k)}^n \frac{(1 - abq^{2m+1})(aq, abq; q)_m}{(1 - abq)(q, bq; q)_m (aq^{j+k+1})^m} \\
& \times \begin{bmatrix} m \\ j \end{bmatrix}_q \cdot \begin{bmatrix} m \\ k \end{bmatrix}_q (abq^{m+1}; q)_j (abq^{m+1}; q)_k.
\end{aligned}$$

*Remark 2.2.* Let  $a = b = 1$  in (2.8), then take  $q \rightarrow 1$  we would get the classical Hilbert matrix.

**Theorem 2.3.** For  $n \in \mathbb{N} \cup \{0\}$  and  $q \neq 0$ , the matrix

$$(2.12) \quad \left( (q^{\alpha+1}; q)_{j+k} q^{-\binom{j+k}{2}} \right)_{j,k=0}^n$$

has determinant

$$(2.13) \quad \det \left( (q^{\alpha+1}; q)_{j+k} q^{-\binom{j+k}{2}} \right)_{j,k=0}^n = \frac{\prod_{k=1}^n (q, q^{\alpha+1}; q)_k}{q^{n(n+1)(4n-1)/6}}.$$

The matrix (2.12) is invertible for

$$(2.14) \quad -\alpha \notin \mathbb{N}, \quad q^k \neq 1, k \in \mathbb{N},$$

the inverse matrix  $(\gamma_{jk})_{j,k=0}^n$  has elements

$$(2.15) \quad \gamma_{jk} = \frac{q^{[j(j-1)+k(k-1)]}}{(q^{\alpha+1}; q)_j (q^{\alpha+1}; q)_k} \sum_{m=\max(j,k)}^n \frac{(q^{\alpha+1}; q)_m}{(q; q)_m} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q q^m.$$

In our last result, we use the following notations

$$(2.16) \quad x = [x] + \{x\}, \quad x \in \mathbb{R}, \quad [x] \in \mathbb{Z}, \quad \{x\} \in [0, 1).$$

**Theorem 2.4.** For  $n \in \mathbb{N} \cup \{0\}$  and  $q \neq 0$ , the matrix

$$(2.17) \quad \left( \frac{[1 + (-1)^{j+k}]}{2} q^{-(j+k)^2/4} (q; q^2)_{\lfloor \frac{j+k}{2} \rfloor} \right)_{j,k=0}^n$$

has the determinant

$$(2.18) \quad \det \left( \frac{[1 + (-1)^{j+k}]}{2} q^{-(j+k)^2/4} (q; q^2)_{\lfloor \frac{j+k}{2} \rfloor} \right)_{j,k=0}^n = \frac{\prod_{k=0}^n (q; q)_k}{q^{n(n+1)(2n+1)/6}}.$$

For

$$(2.19) \quad q^k \neq 1, k \in \mathbb{N},$$

the matrix (2.17) is invertible and has inverse

$$(2.20) \quad \left( \sum_{m=\max(j,k)}^n \frac{\begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q (q, q^2)_{\lfloor \frac{m-j}{2} \rfloor} (q, q^2)_{\lfloor \frac{m-k}{2} \rfloor} \cos \frac{(m-j)\pi}{2} \cos \frac{(m-k)\pi}{2}}{(-1)^m (q; q)_m q^{-m - \binom{j}{2} - \binom{k}{2}}} \right)_{j,k=0}^n.$$

### 3. PROOFS

Given a polynomial  $f(x)$ , we define the  $q$ -difference operator as [1, 2]

$$(3.1) \quad (D_q f)(x) := \frac{f(x) - f(qx)}{(1-q)x}.$$

Let  $w_k(x) := x^k$  for  $k = 0, 1, \dots$  then we have

$$(3.2) \quad D_q w_n(x) = \frac{1 - q^n}{1 - q} w_{n-1}(x),$$

and

$$(3.3) \quad [D_q^k w_n(x)]_{x=0} = \frac{(q; q)_n}{(1-q)^n} \delta_{kn}.$$

For this polynomial sequence we take

$$(3.4) \quad u_k(f(x)) := v_k(f(x)) := \frac{(1-q)^k}{(q; q)_k} [(D_q^k f)(x)]_{x=0},$$

where  $f(x)$  is a polynomial in variable  $x$ .

**3.1. Proof for Theorem 2.1.** The little  $q$ -Jacobi polynomials  $\{p_n(x; a, b|q)\}_{n=0}^\infty$  are defined as [1, 2]

$$(3.5) \quad p_n(x; a, b|q) := {}_2\phi_1 \left( \begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q; qx \right), \quad 0 < q, aq, bq < 1$$

for  $n \geq 0$ , and we assume that

$$(3.6) \quad p_{-1}(x; a, b|q) := 0, \quad p_0(x; a, b|q) := 1.$$

The little  $q$ -Jacobi polynomials  $\{p_n(x; a, b|q)\}_{n=0}^\infty$  have the orthogonal relation

$$(3.7) \quad \sum_{k=0}^\infty \frac{(bq; q)_k (aq)^k}{(q; q)_k} p_m(q^k; a, b|q) p_n(q^k; a, b|q) = h_n(a, b|q) \delta_{mn}$$

for  $m, n \geq 0$  with

$$(3.8) \quad h_n(a, b|q) := \frac{(abq^2; q)_\infty (1-abq)(aq)^n}{(aq; q)_\infty (1-abq^{2n+1})} \frac{(q, bq; q)_n}{(aq, abq; q)_n}.$$

The little  $q$ -Jacobi polynomials satisfy the following difference relation

$$(3.9) \quad D_q^k p_n(x; a, b|q) = \frac{q^k (q^{-n}; q)_k (abq^{n+1}; q)_k}{(1-q)^k (aq; q)_k} p_{n-k}(x; aq^k, bq^k|q),$$

for  $n, k = 0, 1, \dots$  and  $n \geq k$ . The orthonormal polynomial

$$(3.10) \quad p_n(x) = \frac{(-1)^n p_n(x; aq, bq|q)}{\sqrt{h_n(a, b|q)}}$$

have the leading coefficient

$$(3.11) \quad \gamma_n = \sqrt{\frac{(aq; q)_\infty (abq, abq^2, abq^2, abq^3; q^2)_n}{(abq^2; q)_\infty (q, aq, bq, abq; q)_n a^n q^{n^2}}}$$

for  $n = 0, 1, \dots$

The moments are given by the formula

$$(3.12) \quad \mu_n = \sum_{m=0}^{\infty} \frac{(bq; q)_m (aq)^m q^{nm}}{(q; q)_m},$$

or

$$(3.13) \quad \mu_n = \frac{(abq^{n+2}; q)_\infty}{(aq^{n+1}; q)_\infty}.$$

by using the  $q$ -binomial theorem.

For any  $n = 0, 1, \dots$ , the matrix

$$(3.14) \quad H_n = \left( \frac{(abq^{j+k+2}; q)_\infty}{(aq^{j+k+1}; q)_\infty} \right)_{j,k=0}^n$$

has the determinant

$$(3.15) \quad \det \left( \frac{(abq^{j+k+2}; q)_\infty}{(aq^{j+k+1}; q)_\infty} \right)_{j,k=0}^n = \left( \frac{(abq^2; q)_\infty a^{n/2} q^{n(2n+1)/6}}{(aq; q)_\infty} \right)^{n+1} \\ \times \prod_{k=0}^n \frac{(q, aq, bq, abq; q)_k}{(abq, abq^2, abq^2, abq^3; q^2)_k},$$

or

$$(3.16) \quad \det \left( \frac{(aq; q)_{j+k}}{(abq^2; q)_{j+k}} \right)_{j,k=0}^n = a^{n(n+1)/2} q^{n(n+1)(2n+1)/6} \\ \times \prod_{k=1}^n \frac{(q, aq, bq, abq; q)_k}{(abq, abq^2, abq^2, abq^3; q^2)_k}$$

by applying Theorem 1.1.

Under the condition  $0 < q, aq, bq < 1$ , the elements of  $H_n^{-1} = (\beta_{jk})_{0 \leq j, k \leq n}$  are given by

$$(3.17) \quad \beta_{jk} = \frac{q^{j+k} (aq; q)_\infty}{(abq^2; q)_\infty} \sum_{m=\max(j,k)}^n \frac{(1 - abq^{2m+1}) (aq, abq; q)_m}{(1 - abq) (aq)^m (q, bq; q)_m} \\ \times \frac{(q^{-m}; q)_j (abq^{m+1}; q)_j}{(q; q)_j (aq; q)_j} \frac{(q^{-m}; q)_k (abq^{m+1}; q)_k}{(q; q)_k (aq; q)_k}$$

for  $j, k = 0, 1, \dots, n$ . Then, the matrix

$$(3.18) \quad \left( \frac{(aq; q)_{j+k}}{(abq^2; q)_{j+k}} \right)_{j,k=0}^n$$



has the inverse matrix  $(\gamma_{jk})_{j,k=0}^n$  with

$$(3.19) \quad \begin{aligned} \gamma_{jk} &= \frac{(-1)^{j+k} q^{[j(j+1)+k(k+1)]/2}}{(aq; q)_j (aq; q)_k} \\ &\times \sum_{m=\max(j,k)}^n \frac{(1 - abq^{2m+1})(aq, abq; q)_m}{(1 - abq)(q, bq; q)_m (aq^{j+k+1})^m} \\ &\times \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q (abq^{m+1}; q)_j (abq^{m+1}; q)_k. \end{aligned}$$

Since the formulas (3.16), (3.18) and (3.19) contain only rational functions of  $a$ ,  $b$  and  $q^k$ ,  $k \in \mathbb{N}$ , the original restrictions in (3.5) could be dropped and Theorem 2.1 follows.

**3.2. Proof for Theorem 2.3.** The  $q$ -Laguerre polynomials  $\{L_n^{(\alpha)}(x; q)\}_{n=0}^{\infty}$  are defined as [1, 2]

$$(3.20) \quad L_n^{(\alpha)}(x; q) := \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^n \frac{(q^{-n}; q)_k q^{\binom{k+1}{2}} (-x)^k q^{(\alpha+n)k}}{(q; q)_k (q^{\alpha+1}; q)_k}, \quad q \in (0, 1), \alpha > -1$$

for  $n \geq 0$ , and we assume that

$$(3.21) \quad L_{-1}^{(\alpha)}(x; q) := 0, \quad L_0^{(\alpha)}(x; q) := 1.$$

The  $q$ -Laguerre polynomials have an orthogonal relation

$$(3.22) \quad \sum_{k=-\infty}^{\infty} \frac{q^{k\alpha+k}}{(-cq^k; q)_{\infty}} L_m^{(\alpha)}(cq^k; q) L_n^{(\alpha)}(cq^k; q) = h_n(c, \alpha|q) \delta_{mn}$$

for  $m, n = 0, 1, \dots$  with

$$(3.23) \quad h_n(c, \alpha|q) := \frac{(q, -cq^{\alpha+1}, -c^{-1}q^{-\alpha}; q)_{\infty}}{(q^{\alpha+1}, -c, -c^{-1}q; q)_{\infty}} \frac{(q^{\alpha+1}; q)_n}{(q; q)_n q^n}.$$

The  $q$ -Laguerre polynomials satisfy the following difference relation

$$(3.24) \quad D_q^k L_n^{(\alpha)}(x; q) = \frac{q^{k\alpha+k^2}}{(1-q)^k} L_{n-k}^{(\alpha+k)}(q^k x; q)$$

for  $n, k = 0, 1, \dots$ . The orthonormal  $q$ -Laguerre polynomial

$$(3.25) \quad \ell_n(x) := \frac{L_n^{(\alpha)}(x; q)}{\sqrt{h_n(c, \alpha|q)}}$$

has the leading coefficient

$$(3.26) \quad \gamma_n = \frac{q^{(\alpha+n+1/2)n}}{\sqrt{(q, q^{\alpha+1}; q)_n}} \sqrt{\frac{(q^{\alpha+1}, -c, -c^{-1}q; q)_{\infty}}{(q, -cq^{\alpha+1}, -c^{-1}q^{-\alpha}; q)_{\infty}}}$$

for  $n = 0, 1, \dots$

The moments are given by

$$(3.27) \quad \begin{aligned} \mu_n &= \sum_{k=-\infty}^{\infty} \frac{c^n q^{k(\alpha+n+1)}}{(-cq^k; q)_{\infty}} \\ &= \frac{c^n}{(-c; q)_{\infty}} \sum_{k=-\infty}^{\infty} (-c; q)_k q^{k(\alpha+n+1)}, \end{aligned}$$

which could be summed by the formula (2.7),

$$(3.28) \quad \begin{aligned} \mu_n &= \frac{c^n}{(-c; q)_{\infty}} \frac{(q, -cq^{\alpha+n+1}, -q^{-\alpha-n}/c; q)_{\infty}}{(-q/c, q^{\alpha+n+1}; q)_{\infty}} \\ &= \frac{c^n (q^{\alpha+1}; q)_{\infty} (-q^{-\alpha-n}/c; q)_{\infty}}{(-cq^{\alpha+1}; q)_{\infty}} \frac{(q, -cq^{\alpha+1}, -q^{-\alpha}/c; q)_{\infty}}{(-c, -q/c, q^{\alpha+1}; q)_{\infty}} \\ &= \frac{(q^{\alpha+1}; q)_{\infty}}{q^{n\alpha+n(n+1)/2}} \frac{(q, -cq^{\alpha+1}, -q^{-\alpha}/c; q)_{\infty}}{(-c, -q/c, q^{\alpha+1}; q)_{\infty}}. \end{aligned}$$

Then for any  $n = 0, 1, \dots$ , the matrix

$$(3.29) \quad H_n = \left( \frac{(q^{\alpha+1}; q)_{j+k}}{q^{[(j+k)\alpha+(j+k)(j+k+1)/2]}} \frac{(q, -cq^{\alpha+1}, -q^{-\alpha}/c; q)_{\infty}}{(-c, -q/c, q^{\alpha+1}; q)_{\infty}} \right)_{j,k=0}^n$$

has the determinant

$$(3.30) \quad \det H_n = \left( \frac{(q, -cq^{\alpha+1}, -c^{-1}q^{-\alpha}; q)_{\infty}}{(q^{\alpha+1}, -c, -c^{-1}q; q)_{\infty}} \right)^{n+1} \prod_{k=0}^n \frac{(q, q^{\alpha+1}; q)_k}{q^{k\alpha+k(k+1)/2}},$$

or

$$(3.31) \quad \det \left( (q^{\alpha+1}; q)_{j+k} q^{-\binom{j+k}{2}} \right)_{j,k=0}^n = \frac{\prod_{k=1}^n (q, q^{\alpha+1}; q)_k}{q^{n(n+1)(4n-1)/6}},$$

by applying Theorem 1.1. By applying Corollary ??, we obtain the inverse matrix  $H_n^{-1} = (\beta_{jk})_{j,k=0}^n$  with

$$(3.32) \quad \begin{aligned} \beta_{jk} &= \frac{(q^{\alpha+1}, -c, -c^{-1}q; q)_{\infty}}{(q, -cq^{\alpha+1}, -c^{-1}q^{-\alpha}; q)_{\infty}} \frac{q^{\alpha(j+k)+j^2+k^2}}{(q; q)_j (q; q)_k} \\ &\times \sum_{m=\max(j,k)}^n \frac{(q; q)_m q^m}{(q^{\alpha+1}; q)_m} \frac{(q^{\alpha+j+1}; q)_{m-j} (q^{\alpha+k+1}; q)_{m-k}}{(q; q)_{m-j} (q; q)_{m-k}}, \end{aligned}$$

then, the inverse matrix  $(\gamma_{jk})_{j,k=0}^n$  of

$$(3.33) \quad \left( (q^{\alpha+1}; q)_{j+k} q^{-\binom{j+k}{2}} \right)_{j,k=0}^n$$

has element

$$(3.34) \quad \begin{aligned} \gamma_{jk} &= \frac{q^{j^2-j+k^2-k}}{(q; q)_j (q; q)_k} \sum_{m=\max(j,k)}^n \frac{(q; q)_m q^m}{(q^{\alpha+1}; q)_m} \frac{(q^{\alpha+1}; q)_{m-j} (q^{\alpha+1}; q)_{m-k}}{(q; q)_{m-j} (q; q)_{m-k}} \\ &= \frac{q^{[j(j-1)+k(k-1)]}}{(q^{\alpha+1}; q)_j (q^{\alpha+1}; q)_k} \sum_{m=\max(j,k)}^n \frac{(q^{\alpha+1}; q)_m q^m}{(q; q)_m} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q, \end{aligned}$$

or

$$(3.35) \quad \gamma_{jk} = \frac{q^{[j(j-1)+k(k-1)]}}{(q^{\alpha+1}; q)_j (q^{\alpha+1}; q)_k} \sum_{m=\max(j,k)}^n \frac{(q^{\alpha+1}; q)_m q^m}{(q; q)_m} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q.$$

Since the formulas (3.31), (3.33) and (3.35) only involve rational functions of  $q^\alpha$  and  $q^k, k \in \mathbb{N}$ , thus the restriction  $q$  may be dropped to get Theorem 2.3.

**3.3. Proof for Theorem 2.4.** The discrete  $q$ -Hermite polynomials II are defined as [2]

$$(3.36) \quad \tilde{h}_n(x; q) := x^n {}_2\phi_n \left( \begin{matrix} q^{-n}, q^{-n+1} \\ 0 \end{matrix}; q^2; -\frac{q^2}{x^2} \right)$$

for  $n \geq 0$  and we assume that

$$(3.37) \quad \tilde{h}_{-1}(x; q) := 0, \quad \tilde{h}_0(x; q) := 1.$$

The polynomials  $\{\tilde{h}_n(x; q)\}_{n=0}^\infty$  satisfy the difference equation

$$(3.38) \quad D_{q^{-1}}^j \tilde{h}_n(x; q) = \frac{(q; q)_n q^{-jn+j(j+1)/2}}{(1-q)^j (q; q)_{n-j}} \tilde{h}_{n-j}(x; q).$$

They also satisfy the orthogonality

$$(3.39) \quad \sum_{k=-\infty}^{\infty} \left[ \tilde{h}_m(cq^k; q) \tilde{h}_n(cq^k; q) + \tilde{h}_m(-cq^k; q) \tilde{h}_n(-cq^k; q) \right] w(cq^k) q^k = h_n(c|q) \delta_{mn}$$

for  $m, n \geq 0$ , where

$$(3.40) \quad w(x) := \frac{1}{(-x^2; q^2)_\infty},$$

and

$$(3.41) \quad h_n(c|q) := 2 \frac{(q^2, -c^2q, -c^{-2}q; q^2)_\infty}{(q, -c^2, -c^{-2}q^2; q^2)_\infty} \frac{(q; q)_n}{q^{n^2}}$$

for some  $c > 0$ . thus the orthonormal polynomial

$$(3.42) \quad h_n(x) := \frac{\tilde{h}_n(x; q)}{\sqrt{h_n(c|q)}}$$

has the leading coefficient

$$(3.43) \quad \gamma_n = \frac{1}{\sqrt{h_n(c|q)}}.$$

The moments of the measure could be calculated via formula 2.7,

$$\begin{aligned}
 (3.44) \quad \mu_n &= \sum_{k=-\infty}^{\infty} [(cq^k)^n + (-cq^k)^n] w(cq^k) q^k \\
 &= [1 + (-1)^n] c^n \sum_{k=-\infty}^{\infty} \frac{q^{(n+1)k}}{(-c^2 q^{2k}; q^2)_{\infty}} \\
 &= \frac{[1 + (-1)^n] c^n}{(-c^2; q^2)_{\infty}} \sum_{k=-\infty}^{\infty} (-c^2; q^2)_k q^{(n+1)k} \\
 &= \frac{[1 + (-1)^n] c^n}{(-c^2; q^2)_{\infty}} \frac{(q^2, -c^{-2} q^{-n+1}, -c^2 q^{n+1}; q^2)_{\infty}}{(-c^{-2} q^2, q^{n+1}; q^2)_{\infty}} \\
 &= \frac{[1 + (-1)^n] q^{-n^2/4} (q^2, -c^{-2} q, -c^2 q; q^2)_{\infty}}{(-c^2, -c^{-2} q^2, q^{n+1}; q^2)_{\infty}}.
 \end{aligned}$$

Thus, for  $n = 0, 1, \dots$ , the matrix

$$(3.45) \quad H_n = \left( \frac{[1 + (-1)^{j+k}] q^{-(j+k)^2/4} (q^2, -c^{-2} q, -c^2 q; q^2)_{\infty}}{(-c^2, -c^{-2} q^2, q^{j+k+1}; q^2)_{\infty}} \right)_{j,k=0}^n$$

has determinant

$$(3.46) \quad \det H_n = \frac{\prod_{k=0}^n (q; q)_k}{q^{n(n+1)(2n+1)/6}} \left[ 2 \frac{(q^2, -c^2 q, -c^{-2} q; q^2)_{\infty}}{(q, -c^2, -c^{-2} q^2; q^2)_{\infty}} \right]^{n+1}.$$

Therefore,

$$(3.47) \quad \det \left( \frac{[1 + (-1)^{j+k}]}{2} q^{-(j+k)^2/4} (q; q^2)_{\lfloor \frac{j+k}{2} \rfloor} \right)_{j,k=0}^n = \frac{\prod_{k=0}^n (q; q)_k}{q^{n(n+1)(2n+1)/6}}.$$

For  $n = 0, 1, \dots$ , the matrix  $H_n^{-1} = (\beta_{jk})_{j,k=0}^n$  has element

$$\begin{aligned}
 (3.48) \quad \beta_{jk} &= (-1)^{j+k} q^{j^2+k^2} \frac{(q, -c^2, -c^{-2} q^2; q^2)_{\infty}}{2(q^2, -c^2 q, -c^{-2} q; q^2)_{\infty}} \\
 &\quad \times \sum_{m=\max(j,k)}^n \frac{q^{m^2-(j+k)m} \tilde{h}_{m-j}(0; q) \tilde{h}_{m-k}(0; q)}{(q; q)_m} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q.
 \end{aligned}$$

From the generating function

$$(3.49) \quad \frac{(-xt; q)_{\infty}}{(-t^2; q^2)_{\infty}} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q; q)_n} \tilde{h}_n(x; q),$$

we get

$$(3.50) \quad \tilde{h}_n(0; q) = \frac{(q; q^2)_{\infty} \cos \frac{n\pi}{2}}{(q^{1+n}; q^2)_{\infty} q^{\binom{n}{2}}}.$$

Then, the matrix

$$(3.51) \quad \left( \frac{[1 + (-1)^{j+k}]}{2} q^{-(j+k)^2/4} (q; q^2)_{\lfloor \frac{j+k}{2} \rfloor} \right)_{j,k=0}^n$$

has inverse matrix

$$(3.52) \quad \left( \sum_{m=\max(j,k)}^n \frac{\begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q (q, q^2)_{\lfloor \frac{m-j}{2} \rfloor} (q, q^2)_{\lfloor \frac{m-k}{2} \rfloor} \cos \frac{(m-j)\pi}{2} \cos \frac{(m-k)\pi}{2}}{(-1)^m (q; q)_m q^{-m - \binom{j}{2} - \binom{k}{2}}} \right)_{j,k=0}^n.$$

Since formulas (3.47), (3.51) and (3.52) contain only rational functions of  $q^k$ ,  $k \in \mathbb{N}$ , thus the restriction may be dropped and Theorem 2.4 follows.

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